

Monopole triangle over integers

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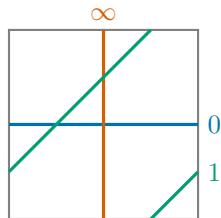
Surgery triad

Definition

Let M be a compact oriented 3-manifold with $\partial M \cong T^2$. Choose oriented curves $\gamma_i \subset \partial M$, $i \in \mathbb{Z}/3$, with

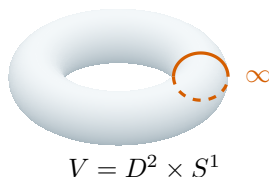
$$\gamma_0 \cdot \gamma_1 = \gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_0 = -1.$$

The fillings $Y_i = M(\gamma_i)$ form a **surgery triad**, where the meridian of the attached solid torus is identified with γ_i . Let $W_i : Y_i \rightarrow Y_{i+1}$ be the corresponding 2-handle cobordism.



∂M

gluing
←



$V = D^2 \times S^1$

Surgery exact triangle

Definition

Let F assign graded modules to closed oriented 3-manifolds and module morphisms to cobordisms. It satisfies the **surgery exact-triangle property** if every surgery triad gives an exact triangle

$$F(Y_0) \xrightarrow{F(W_0)} F(Y_1) \xrightarrow{F(W_1)} F(Y_2) \xrightarrow{F(W_2)} F(Y_0).$$

That is, for $i \in \mathbb{Z}/3$, $\text{im } F(W_i) = \ker F(W_{i+1})$.

Example (Unknot)

$$\begin{array}{ccccc} \infty & & 0 & & 1 \\ \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc \\ S^3 & & S^1 \times S^2 & & S^3 \end{array} \longrightarrow S^3$$

In Heegaard Floer homology over $\mathbb{F}_2 = \mathbb{Z}/2$,

$$\mathbb{F}_2 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathbb{F}_2^2 \xrightarrow{(0,1)} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2.$$

Surgery generation

Theorem (Baldwin–Bloom, Culler '14, Ni '25)

Closed oriented connected 3-manifolds are generated by S^3 using two members of a surgery triad $\Rightarrow$ the third.

Corollary (used by Lin '17)

Let F, G be functors from the cobordism category to the same category of graded modules that send surgery triads to exact triangles. Suppose $\Psi : F \Rightarrow G$ is a natural transformation. If $\Psi(S^3)$ is an isomorphism, then $\Psi(Y)$ is an isomorphism for every closed oriented connected 3-manifold Y .

Proof. Five lemma and induction.

$$\begin{array}{ccccccccc} F(Y_0) & \rightarrow & F(Y_1) & \rightarrow & F(Y_2) & \rightarrow & F(Y_0) & \rightarrow & F(Y_1) \\ \cong \downarrow & & \cong \downarrow & & \downarrow \Psi(Y_2) & & \cong \downarrow & & \cong \downarrow \\ G(Y_0) & \rightarrow & G(Y_1) & \rightarrow & G(Y_2) & \rightarrow & G(Y_0) & \rightarrow & G(Y_1) \end{array}$$

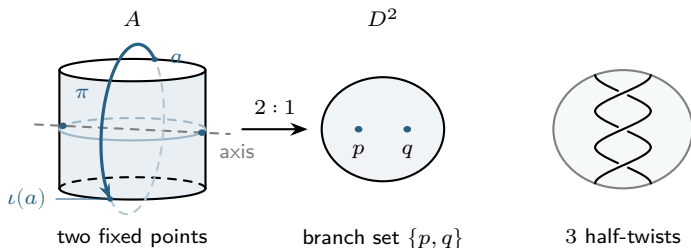
Montesinos trick

Definition

A **rational tangle** is a pair (B^3, τ) homeomorphic to $(D^2 \times I, \{p, q\} \times I)$, where $p, q \in \text{int } D^2$ are distinct. The annulus $A = S^1 \times [-1, 1]$ has involution

$$\iota(e^{i\theta}, t) = (e^{-i\theta}, -t).$$

Its quotient is D^2 , with two branch points. Thus $A \times I \cong S^1 \times D^2 \rightarrow D^2 \times I$ is branched over two trivial arcs.



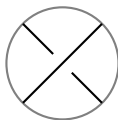
Montesinos trick

Proposition (Montesinos)

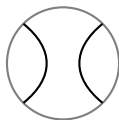
Rational tangle replacement lifts to Dehn filling. For the displayed crossing of L with the two **resolutions** L_0, L_1 ,

$$(-\Sigma_2(L), -\Sigma_2(L_0), -\Sigma_2(L_1))$$

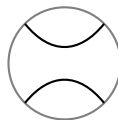
is a surgery triad.



L



L_0



L_1

Floer exact triangles

Theory	Coefficients	Authors
Instanton	$\mathbb{Z}$	Floer '90
Heegaard Floer	$\mathbb{F}_2, \mathbb{Z}$	Ozsváth–Szabó '04
Monopole	$\mathbb{F}_2$	Kronheimer–Mrowka–Ozsváth–Szabó '07
Monopole	$\mathbb{Z}[\sqrt{-1}]$	Freeman '21, with local coefficients
Monopole	$\mathbb{Q}$	Lin–Ruberman–Saveliev '23
Monopole spectra ($b_1(Y_i) = 0$)	$\mathbb{Z}$	Sasahira–Stoffregen (in preparation), mentioned in Dai–Sasahira–Stoffregen '26
Heegaard Floer	$\mathbb{Z}$	Abouzaid–Manolescu '25, with two twisted cobordism maps

Remark

Ozsváth–Szabó and Lin–Ruberman–Saveliev choose ad hoc signs for two cobordism maps. Freeman and Abouzaid–Manolescu use local systems / twisted coefficients, better suited to iteration over a link.

Khovanov-to-Floer spectral sequences

For a diagram with n crossings, let L_v be the resolution at $v \in \{0, 1\}^n$. Iterating the triangle gives

$$E_1 = \bigoplus_{v \in \{0, 1\}^n} F(-\Sigma_2(L_v)), \quad E_2 = H(E_1, d_1).$$

E_2	Target	Coefficients	Authors
$Kh^{\text{red}}(L)$	$\widehat{HF}(-\Sigma_2(L))$	$\mathbb{F}_2$	Ozsváth–Szabó '05
$Kh^{\text{red}}(L)$	$\widehat{HM}(-\Sigma_2(L))$	$\mathbb{F}_2$	Bloom '11
$Kh_{\text{odd}}^{\text{red}}(L)$	$I^\sharp(-\Sigma_2(L))$	$\mathbb{Z}$	Scaduto '15

Scaduto uses surgery triangles for twisted $\text{SO}(3)$ bundles.

Odd Khovanov homology

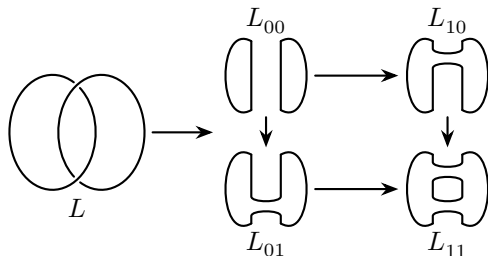
Definition (Ozsváth–Rasmussen–Szabó '13)

For a resolution with k circles, let $V = \mathbb{Z}\langle x_1, \dots, x_k \rangle$. The vertex groups are

even $\mathbb{Z}[x_1, \dots, x_k]/(x_1^2, \dots, x_k^2)$,

odd $\Lambda^* V$.

Note that even and odd Khovanov homology agree over $\mathbb{F}_2$.



For the k -component unlink U_k , we have

$$\Sigma_2(U_k) \cong \#^{k-1}(S^1 \times S^2).$$

Suppressing gradings,

$$Kh^{\text{red}}(U_k; \mathbb{Z}) \cong Kh_{\text{odd}}^{\text{red}}(U_k; \mathbb{Z}) \cong \widehat{HF}(\Sigma_2(U_k); \mathbb{Z}) \cong \Lambda^* \mathbb{Z}^{k-1}.$$

Conjecture (Ozsváth–Rasmussen–Szabó '13)

For any oriented link $L \subset S^3$, there is a spectral sequence

$$Kh_{\text{odd}}^{\text{red}}(L; \mathbb{Z}) \Longrightarrow \widehat{HF}(-\Sigma_2(L); \mathbb{Z}).$$

Main theorem

Theorem (Qiu-Y. '26)

Every surgery triad gives an exact triangle over $\mathbb{Z}$,

$$\begin{array}{ccc} HM(Y_0) & \xrightarrow{HM(W_0)} & HM(Y_1) \\ & \nwarrow HM(W_1; \Gamma_{\nu_1}) & \\ HM(Y_2) & \xrightarrow{HM(W_2; \Gamma_{\nu_2})} & \end{array}$$

after cyclic relabeling, for suitable relative 2-chains ν_1, ν_2 . Here HM is any fixed version among $\widehat{HM}$, $\widetilde{HM}$, $\overline{HM}$, and $\widetilde{\widetilde{HM}}$.

Integral spectral sequence

Theorem (Qiu–Y. '26)

For an oriented link $L \subset S^3$, there is a spectral sequence over $\mathbb{Z}$, possibly depending on a chosen planar diagram of L ,

$$E_2 = Kh_{\text{odd}}^{\text{red}}(L; \mathbb{Z}) \implies \widetilde{HM}(-\Sigma_2(L); \mathbb{Z}).$$

Corollary (Qiu–Y. '26)

The integral isomorphism $\widetilde{HM} \cong \widehat{HF}$ of Kutluhan–Lee–Taubes '10–12 proves the conjecture of Ozsváth–Rasmussen–Szabó. The alternative route $HM \cong ECH \cong HF$ via Taubes '08 and Colin–Ghiggini–Honda '11–12 is proved over $\mathbb{F}_2$, because the second isomorphism uses $\mathbb{F}_2$ coefficients.

Remark

Iteration first gives coefficients Γ_{ν_L} in the target for some integral 1-cycle ν_L . At $u = -1$, these are trivial since

$$[\nu_L] = 0 \in H_1(\Sigma_2(L); \mathbb{Z}/2).$$

Torsion and the choice of coefficients

Example (Shumakovitch '11)

Both $Kh_{\text{odd}}^{\text{red}}(9_{46}; \mathbb{Z})$ and $Kh_{\text{odd}}^{\text{red}}(10_{140}; \mathbb{Z})$ contain 3-torsion.

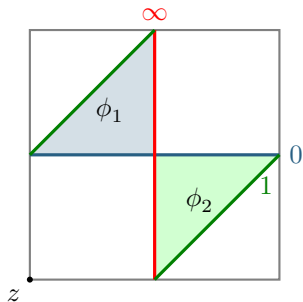
They are thin when 3 is invertible.

Example (Ozsváth–Rasmussen–Szabó '13)

$Kh_{\text{odd}}^{\text{red}}(9_{42}; \mathbb{Q})$ is thin, while $Kh_{\text{odd}}^{\text{red}}(9_{42}; \mathbb{F}_2)$ is not thin.

Here thin means supported in one δ -grading. For any field $\mathbb{F}$, if $Kh_{\text{odd}}^{\text{red}}(L; \mathbb{F})$ is thin, then the spectral sequence collapses at E_2 . The integral spectral sequence retains torsion information.

Necessity of local coefficients



Opposite sides are identified.

Example (Abouzaid–Manolescu '25)

With canonical orientations, the two triangles have the same sign,

$$\epsilon(\phi_1) + \epsilon(\phi_2) = \pm 2.$$

This vanishes over $\mathbb{F}_2$, but not over $\mathbb{Z}$.

Twisting supplies weights $-1, 1$,

$$-\epsilon(\phi_1) + \epsilon(\phi_2) = 0.$$

Remark

An analogous equal-sign obstruction was found in bordered Floer theory by Knowles–Petkova '21. The same failure of cancellation in monopole theory was observed by Freeman '21.

Construction of local coefficients

Let X be a manifold, $f : X \rightarrow \mathbb{R}/\mathbb{Z}$ a continuous map, and $R = \mathbb{Z}[u, u^{-1}]$. Define the local system Γ_u on X with formal rank-one fibers

$$\Gamma_u(x) = u^{f(x)} R.$$

Changing a real lift of $f(x)$ by an integer preserves this module. It is the pullback of the standard local system Γ_{S^1} on $\mathbb{R}/\mathbb{Z}$, with monodromy u . For a real lift $\widetilde{f} \circ \gamma$ along a path γ , transport is multiplication by

$$u^{\widetilde{f} \circ \gamma(1) - \widetilde{f} \circ \gamma(0)}.$$

The monodromy around a loop is $u^{\deg(f \circ \gamma)}$. This defines $H_*(X; \Gamma_u)$.

The above construction is from Kronheimer–Mrowka '11, '17, '19 for instanton homology with local coefficients.

Our construction for monopole theory

Let $\mathcal{B}^\sigma(Y, \mathfrak{s})$ be the blown-up configuration space in monopole theory. For an integral 1-cycle $\eta \subset Y$, fix a unit section $\tau_{\det}(\eta)$ of the determinant line of the Spin^c bundle. Define

$$f : \mathcal{B}^\sigma(Y, \mathfrak{s}) \rightarrow \mathbb{R}/\mathbb{Z}, \quad f(\mathbf{B}) = \frac{1}{2} \frac{i}{2\pi} \int_{\tau_{\det}(\eta)} B^t \pmod{\mathbb{Z}}.$$

The fiber is $u^{f(\mathbf{B})} \mathbb{Z}[u, u^{-1}]$. Gauge transformations change the normalized integral $\frac{i}{2\pi} \int B^t$ by an even integer, so f descends to the quotient.

Kronheimer–Mrowka '07, in *Monopoles and Three-Manifolds*, use the universal exponential ring $\mathbb{Z}[\mathbb{R}]$ and weights $t^{r_\eta(z)}$, where $r_\eta(z) = \frac{i}{2\pi} \int_{[0,1] \times \eta} F_{A_z^t}$. We use $u^{r_\eta(z)/2}$, so roughly $u = t^2$.

Freeman '21 used $t = \sqrt{-1}$; we use $u = -1$, which allows us to work over $\mathbb{Z}$. For a closed oriented surface $\nu \subset W$, the weight is

$$u^{(\langle c_1(\mathfrak{s}), [\nu] \rangle + \nu \cdot \nu + \chi(\nu))/2}.$$

The Spin^c condition gives $\langle c_1(\mathfrak{s}), [\nu] \rangle \equiv \nu \cdot \nu \pmod{2}$; since $\chi(\nu)$ is even, the exponent is integral.

Thanks for your attention.

Slides on floer.wiki/en